Contents 13

Integration

Learning outcomes

In this Workbook you will learn about integration and about some of the common techniques employed to obtain integrals. You will learn that integration is the inverse operation to differentiation and will also appreciate the distinction between a definite and an indefinite integral. You will understand how a definite integral is related to the area under a curve. You will understand how to use the technique of integration by parts to obtain integrals involving the product of functions. You will also learn how to use partial fractions and trigonometric identities in integration.

Basic Concepts of Integration

13.1

Introduction

When a function $f(x)$ is known we can differentiate it to obtain its derivative $\frac{df}{dx}$. The reverse process is to obtain the function $f(x)$ from knowledge of its derivative. This process is called **integration**. Applications of integration are numerous and some of these will be explored in subsequent Sections. First, what is important is to practise basic techniques and learn a variety of methods for integrating functions.

Prerequisites

Before starting this Section you should ...

- thoroughly understand the various techniques of differentiation

Learning Outcomes

On completion you should be able to ...

- evaluate simple integrals by reversing the process of differentiation
- use a table of integrals
- explain the need for a constant of integration when finding indefinite integrals
- use the rules for finding integrals of sums of functions and constant multiples of functions

1. Integration as differentiation in reverse

Suppose we differentiate the function $y = x^2$. We obtain $\dfrac{dy}{dx} = 2x$. Integration reverses this process and we say that the integral of $2x$ is x^2. Pictorially we can regard this as shown in Figure 1:

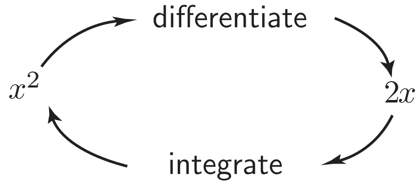


Figure 1

The situation is just a little more complicated because there are lots of functions we can differentiate to give $2x$. Here are some of them: $x^2 + 4$, $x^2 - 15$, $x^2 + 0.5$
All these functions have the same derivative, $2x$, because when we differentiate the constant term we obtain zero. Consequently, when we reverse the process, we have no idea what the original constant term might have been. So we include in our answer an unknown constant, c say, called the **constant of integration**. We state that the integral of $2x$ is $x^2 + c$.

When we want to differentiate a function, $y(x)$, we use the notation $\dfrac{d}{dx}$ as an instruction to differentiate, and write $\dfrac{d}{dx}\big(y(x)\big)$. In a similar way, when we want to integrate a function we use a special notation: $\displaystyle\int y(x)\,dx$.

The symbol for integration, $\displaystyle\int$, is known as an **integral sign**. To integrate $2x$ we write

$$\int \underbrace{2x}\ \mathrm{d}x = x^2 + c$$

integral sign

this term is called the integrand

there must always be a term of the form $\mathrm{d}x$

constant of integration

Note that along with the integral sign there is a term of the form dx, which must always be written, and which indicates the variable involved, in this case x. We say that $2x$ is being **integrated with respect to x**. The function being integrated is called the **integrand**. Technically, integrals of this sort are called **indefinite integrals**, to distinguish them from definite integrals which are dealt with subsequently. When you find an indefinite integral your answer should always contain a constant of integration.

Exercises

1. (a) Write down the derivatives of each of: x^3, $x^3 + 17$, $x^3 - 21$

   (b) Deduce that $\displaystyle\int 3x^2\,dx = x^3 + c$.

2. Explain why, when finding an indefinite integral, a constant of integration is always needed.

Answers

1. (a) $3x^2$, $3x^2$, $3x^2$ (b) Whatever the constant, it is zero when differentiated.

2. Any constant will disappear (i.e. become zero) when differentiated so one must be reintroduced to reverse the
process.

2. A table of integrals

We could use a table of derivatives to find integrals, but the more common ones are usually found in a 'Table of Integrals' such as that shown below. You could check the entries in this table using your knowledge of differentiation. Try this for yourself.

Table 1: Integrals of Common Functions

function $f(x)$	indefinite integral $\int f(x)\,dx$
constant, k	$kx + c$
x	$\frac{1}{2}x^2 + c$
x^2	$\frac{1}{3}x^3 + c$
x^n	$\dfrac{x^{n+1}}{n+1} + c, \quad n \neq -1$
x^{-1} (or $\dfrac{1}{x}$)	$\ln\|x\| + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\cos kx$	$\dfrac{1}{k}\sin kx + c$
$\sin kx$	$-\dfrac{1}{k}\cos kx + c$
$\tan kx$	$\dfrac{1}{k}\ln\|\sec kx\|$+c
e^x	$\mathrm{e}^x + c$
e^{-x}	$-\mathrm{e}^{-x} + c$
e^{kx}	$\dfrac{1}{k}\mathrm{e}^{kx} + c$

When dealing with the trigonometric functions the variable x must always be measured in radians and not degrees. Note that the fourth entry in the Table, for x^n, is valid for any value of n, positive or negative, whole number or fractional, *except* $n = -1$. When $n = -1$ use the fifth entry in the Table.

Example 1

Use Table 1 to find the indefinite integral of x^7: that is, find $\int x^7\,dx$

Solution

From Table 1 note that $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$. In words, this states that to integrate a power of x, increase the power by 1, and then divide the result by the new power. With $n = 7$ we find

$$\int x^7\,dx = \frac{1}{8}x^8 + c$$

Example 2

Find the indefinite integral of $\cos 5x$: that is, find $\int \cos 5x\,dx$

Solution

From Table 1 note that $\int \cos kx\,dx = \dfrac{\sin kx}{k} + c$

With $k = 5$ we find $\int \cos 5x\,dx = \dfrac{1}{5}\sin 5x + c$

In Table 1 the independent variable is always given as x. However, with a little imagination you will be able to use it when other independent variables are involved.

Example 3

Find $\int \cos 5t\,dt$

Solution

We integrated $\cos 5x$ in the previous example. Now the independent variable is t, so simply use Table 1 and replace every x with a t. With $k = 5$ we find

$$\int \cos 5t\,dt = \frac{1}{5}\sin 5t + c$$

It follows immediately that, for example,

$$\int \cos 5\omega\,d\omega = \frac{1}{5}\sin 5\omega + c, \qquad \int \cos 5u\,du = \frac{1}{5}\sin 5u + c \quad \text{and so on.}$$

Example 4

Find the indefinite integral of $\dfrac{1}{x}$: that is, find $\displaystyle\int \frac{1}{x}\,dx$

Solution

This integral deserves special mention. You may be tempted to try to write the integrand as x^{-1} and use the fourth row of Table 1. However, the formula $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ is not valid when $n = -1$ as Table 1 makes clear. This is because we can never divide by zero. Look to the fifth entry of Table 1 and you will see $\displaystyle\int x^{-1}\,dx = \ln|x| + c$.

Example 5

Find $\displaystyle\int 12\,dx$ and $\displaystyle\int 12\,dt$

Solution

In this Example we are integrating a constant, 12. Using Table 1 we find

$$\int 12\,dx = 12x + c \qquad \text{Similarly} \quad \int 12\,dt = 12t + c.$$

Task

Find $\displaystyle\int t^4\,dt$

Your solution

Answer

$$\int t^4\,dt = \frac{1}{5}t^5 + c.$$

Task

Find $\displaystyle\int \frac{1}{x^5}\,dx$ using the laws of indices to write the integrand as x^{-5} and then use Table 1:

Your solution

Answer

$-\dfrac{1}{4}x^{-4} + c = -\dfrac{1}{4x^4} + c.$

Task

Find $\displaystyle\int \mathrm{e}^{-2x}\,dx$ using the entry in Table 1 for integrating e^{kx}:

Your solution

Answer

With $k = -2$, we have $\int \mathrm{e}^{-2x}\,dx = -\dfrac{1}{2}\mathrm{e}^{-2x} + c.$

Exercises

1. Integrate each of the following functions with respect to x:
   (a) x^9, (b) $x^{1/2}$, (c) x^{-3}, (d) $1/x^4$, (e) 4, (f) $\sqrt{x}$, (g) e^{4x}

2. Find (a) $\displaystyle\int t^2\,dt$, (b) $\displaystyle\int_{\text{Text}} 6\,dt$, (c) $\displaystyle\int \sin 3t\,dt$, (d) $\displaystyle\int \mathrm{e}^{7t}\,dt$.

Answers

1 (a) $\dfrac{1}{10}x^{10} + c$, (b) $\dfrac{2}{3}x^{3/2} + c$, (c) $-\dfrac{1}{2}x^{-2} + c$, (d) $-\dfrac{1}{3}x^{-3} + c$, (e) $4x + c$,

(f) same as (b), (g) $\dfrac{1}{4}\mathrm{e}^{4x} + c$

2. (a) $\dfrac{1}{3}t^3 + c$, (b) $6t + c$, (c) $-\dfrac{1}{3}\cos 3t + c$, (d) $\dfrac{1}{7}\mathrm{e}^{7t} + c$

3. Some rules of integration

To enable us to find integrals of a wider range of functions than those normally given in a table of integrals we can make use of the following rules.

The integral of $k\,f(x)$ where k is a constant

A constant factor in an integral can be moved outside the integral sign as follows:

Key Point 1

$$\int k\,f(x)\,dx = k\int f(x)\,dx$$

Example 6

Find the indefinite integral of $11x^2$: that is, find $\int 11x^2\,dx$

Solution

$$\int 11x^2\,dx = 11\int x^2\,dx = 11\left(\frac{x^3}{3}+c\right) = \frac{11x^3}{3}+K \quad \text{where } K \text{ is a constant.}$$

Example 7

Find the indefinite integral of $-5\cos x$; that is, find $\int -5\cos x\,dx$

Solution

$$\int -5\cos x\,dx = -5\int \cos x\,dx = -5\,(\sin x + c) = -5\sin x + K \quad \text{where } K \text{ is a constant.}$$

The integral of $f(x) + g(x)$ and of $f(x) - g(x)$

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

Key Point 2

$$\int [\, f(x) + g(x)\,]\; dx = \int f(x)\, dx + \int g(x)\, dx$$

$$\int [\, f(x) - g(x)\,]\; dx = \int f(x)\, dx - \int g(x)\, dx$$

Example 8

Find $\displaystyle\int (x^3 + \sin x)\, dx$

Solution

$$\int (x^3 + \sin x)\, dx = \int x^3\, dx + \int \sin x\, dx = \frac{1}{4}x^4 - \cos x + c$$

Note that only a single constant of integration is needed.

Task

Find $\displaystyle\int (3t^4 + \sqrt{t})\, dt$

Use Key Points 1 and 2:

Your solution

Answer

$\dfrac{3}{5}t^5 + \dfrac{2}{3}t^{3/2} + c$

Task

The hyperbolic sine and cosine functions, $\sinh x$ and $\cosh x$, are defined as follows:

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \qquad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$$

Note that they are combinations of the exponential functions e^x and e^{-x}.
Find the indefinite integrals of $\sinh x$ and $\cosh x$.

Your solution

$$\int \sinh x\,dx = \int \left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right)\,dx =$$

$$\int \cosh x\,dx = \int \left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)\,dx =$$

Answer

$$\int \sinh x\,dx = \frac{1}{2}\int \mathrm{e}^x\,dx - \frac{1}{2}\int \mathrm{e}^{-x}\,dx = \frac{1}{2}\mathrm{e}^x + \frac{1}{2}\mathrm{e}^{-x} + c = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) + c = \cosh x + c.$$

Similarly $\int \cosh x\,dx = \sinh x + c.$

Further rules for finding more complicated integrals are dealt with in subsequent Sections.

Engineering Example 1

Electrostatic charge

Introduction

Electrostatic charge is important both where it is wanted, as in the electrostatic precipitator plate systems used for cleaning gases, and where it is unwanted, such as when charge builds up on moving belts. This Example is concerned with a charged object with a particular idealised shape - a sphere. However, similar analytical calculations can be carried out for certain other shapes and numerical methods can be used for more complicated shapes.
The electric field at all points inside and outside a charged sphere is given by

$$E(r) = \frac{Qr}{4\pi\varepsilon_0 a^3} \quad \text{if} \quad r < a \tag{1a}$$

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{if} \quad r \geq a \tag{1b}$$

where ε_0 is the permittivity of free space, Q is the total charge, a is the radius of the sphere, and r is the radial distance between the centre of the sphere and a point of observation (see Figure 2).

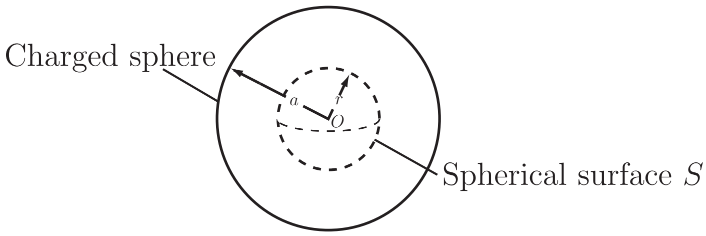


Figure 2: Geometry and symbols associated with the charged sphere

The electric field associated with electrostatic charge has a scalar potential. The electric field defined by (1a) and (1b) shows only a radial dependence of position. Therefore, the electric scalar potential $V(r)$ is related to the field $E(r)$ by

$$E(r) = -\frac{dV}{dr}. \tag{2}$$

Problem in words
A sphere is charged with a uniform density of charge and no other charge is present outside the sphere in space. Determine the variation of electric potential with distance from the centre of the sphere.

Mathematical statement of problem
Determine the electric scalar potential as a function of r, $V(r)$, by integrating (2).

Mathematical analysis
Equation (2) yields $V(r)$ as the negative of the indefinite integral of $E(r)$.

$$-\int dV = \int E(r)\,dr. \tag{3}$$

Using (1a) and (1b) with (3) leads to

$$V(r) = -\frac{Q}{4\pi\varepsilon_0 a^3}\int r\,dr \quad \text{if} \quad r < a \tag{4a}$$

$$V(r) = -\frac{Q}{4\pi\varepsilon_0}\int \frac{dr}{r^2} \quad \text{if} \quad r \geq a \tag{4b}$$

Using the facts that $\int r\,dr = r^2/2 + c_1$ and $\int \frac{dr}{r^2} = -\frac{1}{r} + c_2$,
(4a) and (4b) become

$$V(r) = -\frac{Qr^2}{8\pi\varepsilon_0 a^3} + c_1 \quad \text{if} \quad r < a \tag{5a}$$

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} + c_2 \quad \text{if} \quad r \geq a \tag{5b}$$

The integration constant c_2 can be determined by assuming that the electric potential is zero at an infinite distance from the sphere:

$$\lim_{r\to\infty}[V(r)] = 0 \qquad \Rightarrow \quad \lim_{r\to\infty}\left[-\frac{Q}{4\pi\varepsilon_0 r}\right] + c_2 = 0 \qquad \Rightarrow \quad c_2 = 0.$$

The constant c_1 can be determined by assuming that the potential is continuous at $r = a$.
From equation (5a)

$$V(a) = -\frac{Qa^2}{8\pi\varepsilon_0 a^3} + c_1$$

From equation (5b)

$$V(a) = \frac{Q}{4\pi\varepsilon_0 a}$$

Hence

$$c_1 = \frac{Q}{4\pi\varepsilon_0 a} + \frac{2Q}{8\pi\varepsilon_0 a} = \frac{3Q}{8\pi\varepsilon_0 a}.$$

Substituting for c_1 in (5), the electric potential is obtained for all space is:

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3a^2 - r^2}{2a^3}\right) \quad \text{if} \quad r < a.$$

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} \quad \text{if} \quad r \geq a$$

Interpretation

The potential of the electrostatic field outside a charged sphere varies inversely with distance from the centre of the sphere. Inside the sphere, the electrostatic potential varies with the square of the distance from the centre.

An Engineering Exercise in HELM 29.3 derives the corresponding expressions for the variation of the electrostatic field and an Engineering Exercise in HELM 27.4 calculates the potential energy due to the charged sphere.

Exercises

1. Find $\displaystyle\int (2x - \mathsf{e}^x)\, dx$
2. Find $\displaystyle\int 3\mathsf{e}^{2x}\, dx$
3. Find $\displaystyle\int \frac{1}{3}(x + \cos 2x)\, dx$
4. Find $\displaystyle\int 7x^{-2}\, dx$
5. Find $\displaystyle\int (x+3)^2\, dx$, (be careful!)

Answers

1. $x^2 - \mathsf{e}^x + c$
2. $\dfrac{3}{2}\mathsf{e}^{2x} + c$
3. $\dfrac{1}{6}x^2 + \dfrac{1}{6}\sin 2x + c$
4. $-\dfrac{7}{x} + c$
5. $\dfrac{1}{3}x^3 + 3x^2 + 9x + c$

Definite Integrals 13.2

Introduction

When you were first introduced to integration as the reverse of differentiation, the integrals you dealt with were **indefinite integrals**. The result of finding an indefinite integral is usually a function plus a constant of integration. In this Section we introduce **definite integrals**, so called because the result will be a definite answer, usually a number, with no constant of integration. Definite integrals have many applications, for example in finding areas bounded by curves, and finding volumes of solids.

Prerequisites

Before starting this Section you should ...

- understand integration as the reverse of differentiation
- be able to use a table of integrals

Learning Outcomes

On completion you should be able to ...

- find simple definite integrals
- handle some integrals involving an infinite limit of integration

1. Definite integrals

We saw in the previous Section that $\int f(x)\,dx = F(x) + c$ where $F(x)$ is that function which, when differentiated, gives $f(x)$. That is, $\dfrac{dF}{dx} = f(x)$. For example,

$$\int \sin(3x)\,dx = -\frac{\cos(3x)}{3} + c$$

Here, $f(x) = \sin(3x)$ and $F(x) = -\frac{1}{3}\cos(3x)$ We now consider a definite integral which is simply an indefinite integral but with numbers written to the upper and lower right of the integral sign. The quantity

$$\int_a^b f(x)\,dx$$

is called the definite integral of $f(x)$ from a to b. The numbers a and b are known as the **lower limit** and **upper limit** respectively of the integral. We *define*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

so that a definite integral is usually a number. The *meaning* of a definite integral will be developed in later Sections. For the present we concentrate on the *process* of evaluating definite integrals.

2. Evaluating definite integrals

When you evaluate a definite integral the result will usually be a number. To see how to evaluate a definite integral consider the following Example.

Example 9

Find the definite integral of x^2 from 1 to 4; that is, find $\displaystyle\int_1^4 x^2\,dx$

Solution

$$\int x^2\,dx = \tfrac{1}{3}x^3 + c$$

Here $f(x) = x^2$ and $F(x) = \frac{x^3}{3}$. Thus, according to our definition

$$\int_1^4 x^2\,dx = F(4) - F(1) = \frac{4^3}{3} - \frac{1^3}{3} = 21$$

Writing $F(b) - F(a)$ each time we calculate a definite integral becomes laborious so we replace this difference by the shorthand notation $\Big[\, F(x) \Big]_a^b$. Thus

$$\Big[\, F(x) \Big]_a^b \equiv F(b) - F(a)$$

Thus, from now on, we shall write

$$\int_a^b f(x)\, dx = \Big[\, F(x) \Big]_a^b$$

so that, for example

$$\int_1^4 x^2\, dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = 21$$

Example 10

Find the definite integral of $\cos x$ from 0 to $\dfrac{\pi}{2}$; that is, find $\displaystyle\int_0^{\pi/2} \cos x\, dx$.

Solution

Since $\int \cos x\, dx = \sin x + c$ then

$$\begin{aligned} \int_0^{\pi/2} \cos x\, dx &= \Big[\, \sin x \Big]_0^{\pi/2} \\ &= \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1 - 0 = 1 \end{aligned}$$

Always remember, that if you use a calculator to evaluate any trigonometric functions, you must work in **radian mode**.

Task

Find the definite integral of $x^2 + 1$ from 1 to 2; that is; find $\displaystyle\int_1^2 (x^2 + 1)\, dx$

First perform the integration:

Your solution

Answer

$\left[\dfrac{1}{3}x^3 + x\right]_1^2.$

Now insert the limits of integration, the upper limit first, and hence evaluat the integral:

Your solution

Answer

$\left(\dfrac{8}{3}+2\right)-\left(\dfrac{1}{3}+1\right)=\dfrac{10}{3}$ or 3.333 (3 d.p.).

Task

Find $\displaystyle\int_2^1 (x^2+1)\,dx$.

This Task is very similar to the previous Task. Note the limits have been interchanged:

Your solution

Answer

$\left[\dfrac{1}{3}x^3+x\right]_2^1=\left[\dfrac{1}{3}+1\right]-\left[\dfrac{8}{3}+2\right]=-\dfrac{10}{3}.$

Note from these two Tasks that interchanging the limits of integration, changes the sign of the answer.

Key Point 3

If you interchange the limits, you must change the sign:

$$\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$$

Task

When a spring is fixed at one end and stretched at the free end it exerts a restoring force that is proportional to the displacement of the free end. The constant of proportionality k N m^{-1} is known as the **stiffness** of the spring. Calculate the work done in stretching a spring with stiffness k from displacement x_1 m to displacement x_2 m $(x_2 > x_1)$ given that the work done (W) is the product of force and displacement.

Your solution

Answer

The restoring force varies during the displacement. So the work done during the extension cannot be determined from a single simple product.

Consider a small element Δx of the extension beyond an arbitrary displacement x. The element is sufficiently small that the force during the displacement can be regarded as constant and equal to the force at displacement x is kx. So the work done ΔW in extending the spring from displacement x to displacement $x + \Delta x$ is approximately $kx\Delta x$.

Using the idea of integration as a limit of a sum, in this case as Δx tends to zero,

$$W = \int_{x_1}^{x_2} kx\,dx = \left[\frac{1}{2}kx^2\right]_{x_1}^{x_2} = \frac{1}{2}k(x_2^2 - x_1^2)$$

Exercises

1. Evaluate (a) $\displaystyle\int_0^1 x^2\,dx$, (b) $\displaystyle\int_2^3 \frac{1}{x^2}\,dx$ (c) $\displaystyle\int_1^2 \mathrm{e}^x\,dx$ (d) $\displaystyle\int_{-1}^1 (1+t^2)\,dt$

2. Find (a) $\displaystyle\int_0^{\pi/3} \cos 2x\,dx$ (b) $\displaystyle\int_0^{\pi} \sin x\,dx$ (c) $\displaystyle\int_1^3 \mathrm{e}^{2t}\,dt$

Answers

1. (a) $\dfrac{1}{3}$ (b) $\dfrac{1}{6}$ (c) $\mathrm{e}^2 - \mathrm{e}^1 = 4.671$ (d) 2.667

2 (a) $\sqrt{3}/4 = 0.4330$ (b) 2 (c) 198.019

Engineering Example 2

Torsion of a mild-steel bar

Introduction

For materials such as mild-steel, the relationship between applied shear stress and shear strain (deformation) can be described as follows.

- For small values of the shear strain, the shear stress (τ) and shear strain (ω) are proportional to one another, i.e.

$$\omega = \frac{1}{G} \times \tau \qquad (1)$$

(where G is the *shear modulus*). This is known as **elastic behaviour**.

- There is a maximum shear stress that the material is capable of supporting. If the shear strain is increased further, the shear stress remains roughly constant. This is known as **plastic behaviour**.

Figure 3 summarises the relationship between shear stress and shear strain; the point (ω_Y, τ_Y) is known as the **yield point**.

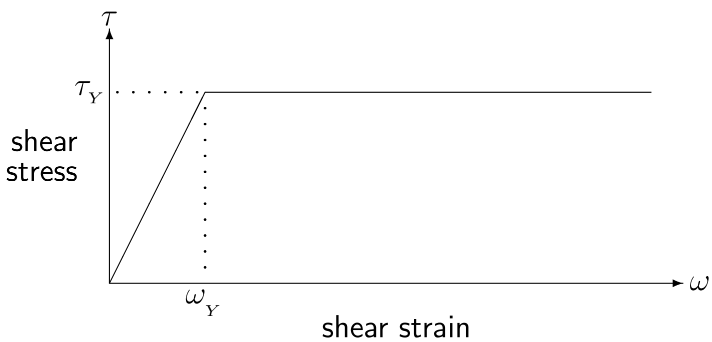


Figure 3

Now suppose that one end of a bar of circular cross section is twisted through an angle θ, then the shear strain on the surface is given by

$$\omega_s = \frac{R\,\theta}{L} \qquad (2)$$

(where R and L are the radius and length of the bar respectively), while the shear strain, at a distance r from the central core, is given by

$$\omega = \frac{r\,\theta}{L} \qquad (3)$$

The torque transmitted by a bar is given by the integral

$$T = \int_0^R 2\pi\, r^2\, \tau(r)\, dr \qquad (4)$$

As the shear strain is a function of distance from the central axis of the bar, it may be that the shear strain on the surface is greater than the critical shear strain ω_Y. In this scenario the shear stress is given by

$$\tau = \begin{cases} \dfrac{\tau_Y}{\omega_Y}\,\omega & \omega \leq \omega_Y \\ \\ \tau_Y & \omega > \omega_Y \end{cases} \tag{5}$$

i.e. the regions near the central axis exhibit elasticity, but in those regions near the surface the elastic limit has been exceeded and the metal exhibits plasticity (see Figure 4).

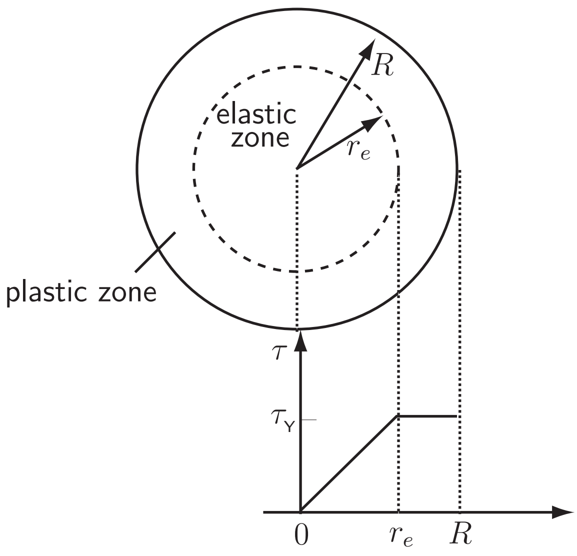


Figure 4

Problem in words

Find an expression for the torque transmitted by a bar as a function of the angle θ through which one end is turned.

Mathematical statement of problem

Using Equations (3) to (5), find a formula for T in terms of the variable θ.

Mathematical analysis

Substituting (3) into (5)

$$\begin{aligned} \tau &= \begin{cases} \dfrac{\tau_Y}{\omega_Y}\dfrac{r\,\theta}{L} & \dfrac{r\,\theta}{L} \leq \omega_Y \\ \\ \tau_Y & \dfrac{r\,\theta}{L} > \omega_Y \end{cases} \\ \\ &= \begin{cases} \dfrac{\tau_Y}{\omega_Y}\dfrac{r\,\theta}{L} & r \leq \dfrac{L\,\omega_Y}{\theta} = r_e \\ \\ \tau_Y & r > \dfrac{L\,\omega_Y}{\theta} = r_e \end{cases} \end{aligned}$$

For small values of θ, $r_e \geq R$ so that the whole of the bar will be in the elastic region, i.e.

$$\tau = \frac{\tau_Y}{\omega_Y}\frac{r\,\theta}{L}$$

Now (4) becomes

$$T = \int_0^R 2\pi\, r^2\, \frac{\tau_Y}{\omega_Y}\frac{r\,\theta}{L}\, dr = 2\pi\, \frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\int_0^R r^3\, dr = 2\pi\,\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\left[\frac{r^4}{4}\right]_0^R = \frac{\pi}{2}\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\,R^4 \qquad (6)$$

i.e. the torque is directly proportional to the twist, θ.

For larger θ, $r_e < R$, so that (4) becomes

$$\begin{aligned}
T &= \int_0^{r_e} 2\pi\, r^2\, \frac{\tau_Y}{\omega_Y}\frac{r\,\theta}{L}\, dr + \int_{r_e}^R 2\pi\, r^2\, \tau_Y\, dr \\
&= 2\pi\,\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\int_0^{r_e} r^3\, dr + 2\pi\,\tau_Y\int_{r_e}^R r^2\, dr \\
&= 2\pi\,\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\left[\frac{r^4}{4}\right]_0^{r_e} + 2\pi\,\tau_Y\left[\frac{r^3}{3}\right]_{r_e}^R \\
&= \frac{\pi}{2}\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\,r_e^4 + \frac{2\pi}{3}\,\tau_Y\left(R^3 - r_e^3\right)
\end{aligned}$$

But $r_e = L\,\omega_Y/\theta$, so

$$\begin{aligned}
T &= \frac{\pi}{2}\frac{\tau_Y}{\omega_Y}\frac{\theta}{L}\frac{L^4\,\omega_Y^4}{\theta^4} + \frac{2\pi}{3}\,\tau_Y\,R^3 - \frac{2\pi}{3}\,\tau_Y\,\frac{L^3\,\omega_Y^3}{\theta^3} \\
&= \frac{2\pi}{3}\,\tau_Y\,R^3 + \pi\left(\frac{1}{2}\,\tau_Y - \frac{2}{3}\,\tau_Y\right)\frac{L^3\,\omega_Y^3}{\theta^3} \\
&= \frac{2\pi}{3}\,\tau_Y\,R^3 - \frac{\pi}{6}\,\tau_Y\,\frac{L^3\,\omega_Y^3}{\theta^3} \qquad (7)
\end{aligned}$$

Equation (6) will apply when $r_e \geq R$, i.e. $(L\,\omega_Y/\theta) \geq R$ or $\theta \leq (L\,\omega_Y/R)$, so that combining (6) and (7) gives overall

$$T = \begin{cases} \dfrac{\pi}{2}\dfrac{\tau_Y}{\omega_Y}\dfrac{\theta}{L}\,R^4 & \theta \leq \dfrac{L\,\omega_Y}{R} \\[2ex] \dfrac{2\pi}{3}\,\tau_Y\,R^3 - \dfrac{\pi}{6}\,\tau_Y\,\dfrac{L^3\,\omega_Y^3}{\theta^3} & \theta > \dfrac{L\,\omega_Y}{R} \end{cases} \qquad (8)$$

Interpretation and further comment

At the critical value of θ, i.e. when the outer edge begins to exhibit plasticity, both formulae in (8) give

$$T_{crit} = \frac{\pi}{2}\,\tau_Y\,R^3$$

Furthermore, the first derivatives are both

$$\frac{dT}{d\theta} = \frac{\pi}{2}\frac{\tau_Y}{\omega_Y}\frac{R^4}{L}$$

i.e. the curves join smoothly.

The second derivatives, though, are not equal (zero in one case). In the theoretical limit as $\theta \to \infty$

$$T = \frac{2\pi}{3}\,\tau_{_Y}\,R^3$$

so this is the total torsional torque which can be carried by the bar. (The critical torque above is three-quarters of this value.) However, clearly $\theta \to \infty$ is merely a theoretical limit since the bar would, in fact, shear at a finite value of θ.

3. Some integrals with infinite limits

On occasions, and notably when dealing with Laplace and Fourier transforms, you will come across integrals in which one of the limits is infinite. We avoid a rigorous treatment of such cases here and instead give some commonly occurring examples.

Example 11

Find the definite integral of e^{-x} from 0 to ∞; that is, find $\displaystyle\int_0^\infty e^{-x}\,dx$.

Solution

The integral is found in the normal way: $\displaystyle\int_0^\infty e^{-x}\,dx = \Big[-e^{-x}\Big]_0^\infty$

There is no difficulty in evaluating the square bracket at the lower limit. We obtain simply $-e^{-0} = -1$. At the upper limit we must examine the behaviour of $-e^{-x}$ as x gets infinitely large. This is where it is important that you are familiar with the properties of the exponential function. If you refer to the graph (Figure 5) you will see that as x tends to infinity e^{-x} tends to zero.

Consequently the contribution to the integral from the upper limit is zero. So

$$\begin{aligned}\int_0^\infty e^{-x}\mathrm{d}x &= \Big[-e^{-x}\Big]_0^\infty \\ &= (-e^{-\infty}) - (-e^{-0}) \\ &= (0) \quad - \quad (-e^{-0}) \\ &= 1\end{aligned}$$

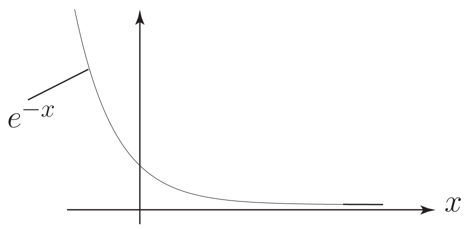


Figure 5

Thus the value of $\displaystyle\int_0^\infty e^{-x}\,dx$ is 1.

Another way of achieving this result is as follows:

We change the infinite limit to a finite limit, b, say and then examine the behaviour of the integral as b tends to infinity, written as

$$\int_0^{\infty} \mathsf{e}^{-x}\,dx = \lim_{b\to\infty}\int_0^b \mathsf{e}^{-x}\,dx$$

Now, $\displaystyle\int_0^b \mathsf{e}^{-x}\,dx = \Big[-\mathsf{e}^{-x}\Big]_0^b = \big(-\mathsf{e}^{-b}\big) - \big(-\mathsf{e}^{-0}\big) = -\mathsf{e}^{-b}+1$

Then as b tends to infinity $-\mathsf{e}^{-b}$ tends to zero, and the resulting integral has the value 1, as before. Many integrals having infinite limits cannot be evaluated in a simple way like this, and many cannot be evaluated at all. Fortunately, most of the integrals you will meet will exhibit the sort of behaviour seen in the last example.

Exercise

Evaluate (a) $\displaystyle\int_1^{\infty} \mathsf{e}^{-x}\,dx$ (b) $\displaystyle\int_0^{\infty} \mathsf{e}^{-2x}\,dx$ (c) $\displaystyle\int_2^{\infty} \mathsf{e}^{-3x}\,dx$ (d) $\displaystyle\int_1^{\infty} \frac{4}{t^2}\,dt$

Answer

(a) $\mathsf{e}^{-1} \sim 0.368$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}\mathsf{e}^{-6} = 0.0008$ (4 d.p.) (d) 4

The Area Bounded by a Curve

13.3

Introduction

One of the important applications of integration is to find the area bounded by a curve. Often such an area can have a physical significance like the work done by a motor, or the distance travelled by a vehicle. In this Section we explain how such an area is calculated.

Prerequisites

Before starting this Section you should ...

- understand integration as the reverse of differentiation
- be able to use a table of integrals
- be able to evaluate definite integrals
- be able to sketch graphs of common functions including polynomials, simple rational functions, exponential functions and trigonometric functions

Learning Outcomes

On completion you should be able to ...

- find the area bounded by a curve and the x-axis
- find the area between two curves

1. Calculating the area under a curve

Let us denote the area under $y = f(x)$ between a fixed point a and a variable point x by $A(x)$:

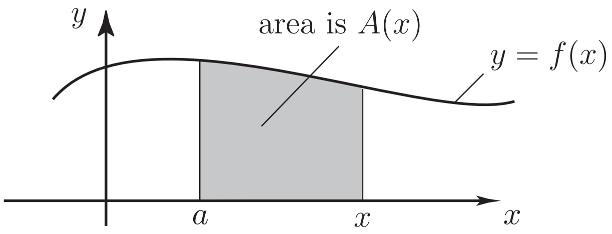


Figure 6

$A(x)$ is clearly a function of x since as the upper limit changes so does the area. How does the area change if we change the upper limit by a very small amount δx? See Figure 7 below.

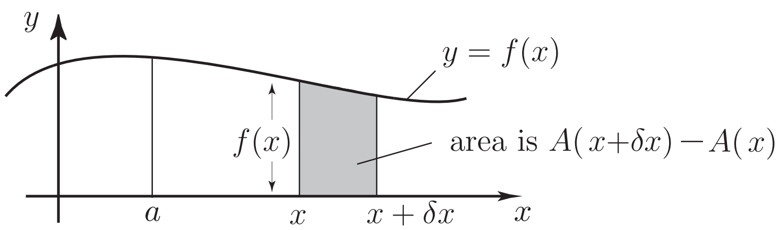


Figure 7

To a good approximation the change in the area is:

$$A(x + \delta x) - A(x) \approx f(x)\delta x$$

[This is because the shaded area is approximately a rectangle with base δx and height $f(x)$.] This approximation gets better and better as δx gets smaller and smaller. Rearranging gives:

$$f(x) \approx \frac{A(x + \delta x) - A(x)}{\delta x}$$

Clearly, in the limit as $\delta x \to 0$ we have

$$f(x) = \lim_{\delta x \to 0} \frac{A(x + \delta x) - A(x)}{\delta x}$$

But this limit on the right-hand side is the **derivative** of $A(x)$ with respect to x so

$$f(x) = \frac{dA(x)}{dx}$$

Thus $A(x)$ is an **indefinite integral** of $f(x)$ and we can therefore write:

$$A(x) = \int f(x)dx$$

Now the area under the curve from a to b is clearly $A(b) - A(a)$. But remembering our shorthand notation for this difference, introduced in the last Section we have, finally

$$A(b) - A(a) \equiv \Big[A(x) \Big]_a^b = \int_a^b f(x)dx$$

We conclude that the area under the curve $y = f(x)$ from a to b is given by the definite integral of $f(x)$ from a to b.

2. The area bounded by a curve lying above the x-axis

Consider the graph of the function $y = f(x)$ shown in Figure 8. Suppose we are interested in calculating the area underneath the graph and above the x-axis, between the points where $x = a$ and $x = b$. When such an area lies entirely above the x-axis, as is clearly the case here, this area is given by the definite integral $\int_a^b f(x)\,dx$.

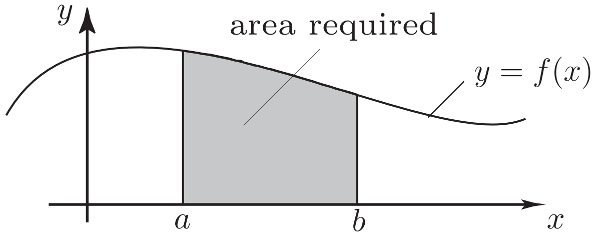


Figure 8

Key Point 4

The area under the curve $y = f(x)$, between $x = a$ and $x = b$ is given by $\displaystyle\int_a^b f(x)\,dx$

when the curve lies entirely above the x-axis between a and b.

Example 12

Calculate the area bounded $y = x^{-1}$ and the x-axis, between $x = 1$ and $x = 4$.

Solution

Below is a graph of $y = x^{-1}$. The area required is shaded; it lies entirely above the x-axis.

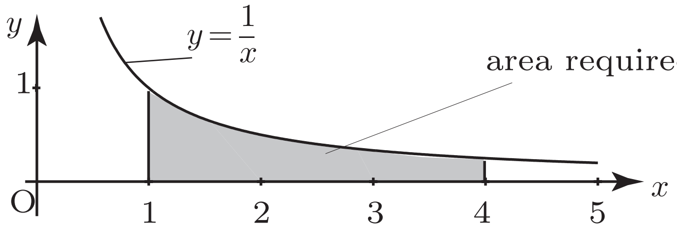
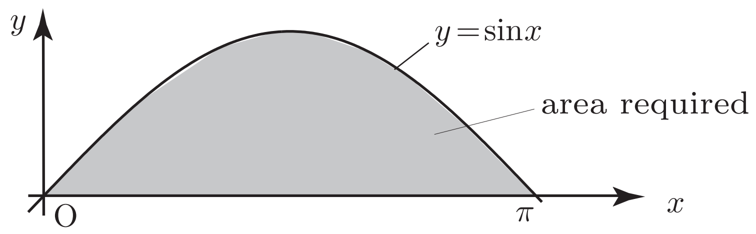


Figure 9

$$\text{area} \quad = \int_1^4 \frac{1}{x} dx = \Big[\ln|x|\Big]_1^4 \quad = \ln 4 - \ln 1 \ = \ln 4 = 1.386 \text{ (3 d.p.)}$$

Task

Find the area bounded by the curve $y = \sin x$ and the x-axis between $x = 0$ and $x = \pi$. (The required area is shown in the figure. Note that it lies entirely above the x-axis.)



Your solution

Answer

$$\int_0^\pi \sin x\, dx = \Big[-\cos x \Big]_0^\pi = 2.$$

Task

Find the area under $f(x) = \mathrm{e}^{2x}$ from $x = 1$ to $x = 3$ given that the exponential function e^{2x} is always positive.

Your solution

Answer

area $= \displaystyle\int_1^3 \mathrm{e}^{2x}\, dx = \left[\frac{1}{2}\mathrm{e}^{2x} \right]_1^3 = 198$ to 3 significant figures.

Example 13

The figure shows the graphs of $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \frac{1}{2}\pi$. The two graphs intersect at the point where $x = \frac{1}{4}\pi$. Find the shaded area.

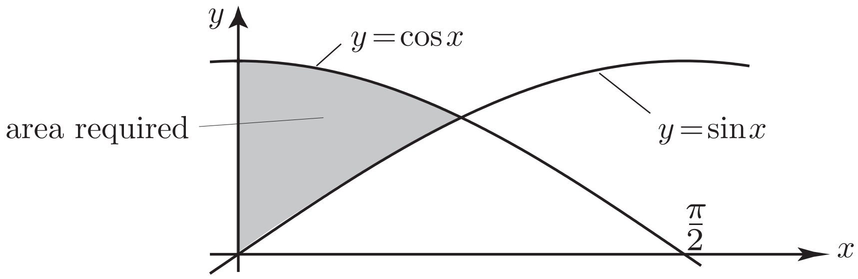


Figure 10

Solution

To find the shaded area we could calculate the area under the graph of $y = \sin x$ for x between 0 and $\frac{1}{4}\pi$, and subtract this from the area under the graph of $y = \cos x$ between the same limits. Alternatively the two processes can be combined into one and we can write

$$\begin{aligned}\text{shaded area} &= \int_0^{\pi/4} (\cos x - \sin x)dx \\ &= \Big[\sin x + \cos x\Big]_0^{\pi/4} \\ &= \left(\sin \tfrac{1}{4}\pi + \cos \tfrac{1}{4}\pi\right) - (\sin 0 + \cos 0) \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1\end{aligned}$$

So the numeric value of the integral is $\frac{2}{\sqrt{2}} - 1 = 0.414$ to 3 d.p.. (Alternatively you can use your calculator to obtain this result directly by evaluating $\sin \dfrac{\pi}{4}$ and $\cos \dfrac{\pi}{4}$.)

Exercises

In each question you should check that the required area lies entirely above the horizontal axis.

1. Find the area under the curve $y = 7x^2$ and above the x-axis between $x = 2$ and $x = 5$.
2. Find the area bounded by the curve $y = x^3$ and the x-axis between $x = 0$ and $x = 2$.
3. Find the area bounded by the curve $y = 3t^2$ and the t-axis between $t = -3$ and $t = 3$.
4. Find the area under $y = x^{-2}$ between $x = 1$ and $x = 10$.

Answer

1. 273, 2. 4, 3. 54, 4. 0.9.

3. The area bounded by a curve, not entirely above the x-axis

Figure 11 shows a graph of $y = -x^2 + 1$.

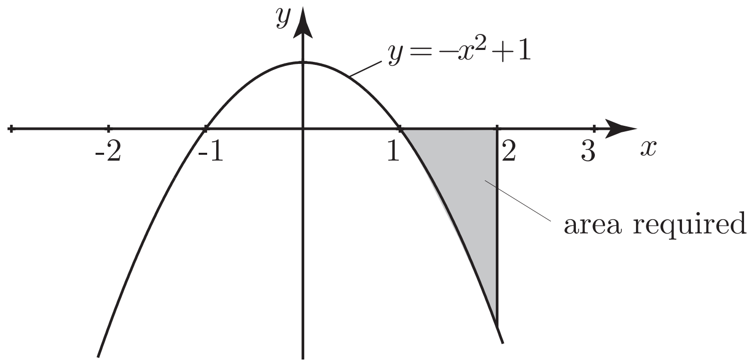


Figure 11

The shaded area is bounded by the x-axis and the curve, but lies entirely below the x-axis. Let us evaluate the integral $\int_1^2 (-x^2 + 1)dx$.

$$\begin{aligned}
\int_1^2 (-x^2+1)dx &= \left[-\frac{x^3}{3} + x\right]_1^2 \\
&= \left(-\frac{2^3}{3} + 2\right) - \left(-\frac{1^3}{3} + 1\right) \\
&= -\frac{7}{3} + 1 = -\frac{4}{3}
\end{aligned}$$

The evaluation of the area yields a negative quantity. There is, of course, no such thing as a negative area. The area is actually $\frac{4}{3}$, and the negative sign is an indication that the area lies below the x-axis. (However, in applications of integration such as work/energy or distance travelled in a given direction negative values can be meaningful.)

If an area contains parts both above and below the horizontal axis, care must be taken when calculating this area. It is necessary to determine which parts of the graph lie above the horizontal axis and which lie below. Separate integrals need to be calculated for each 'piece' of the graph. This idea is illustrated in the next Example.

Example 14

Find the total area enclosed by the curve $y = x^3 - 5x^2 + 4x$ and the x-axis between $x = 0$ and $x = 3$.

Solution

We need to determine which parts of the graph lie above and which lie below the x-axis. To do this it is helpful to consider where the graph cuts the x-axis. So we consider the function $x^3 - 5x^2 + 4x$ and look for its zeros

$$x^3 - 5x^2 + 4x = x(x^2 - 5x + 4) = x(x-1)(x-4)$$

So the graph cuts the x-axis when $x = 0$, $x = 1$ and $x = 4$. Also, when x is large and positive, y is large and positive since the term involving x^3 dominates. When x is large and negative, y is large and negative for the same reason. With this information we can sketch a graph showing the required area:

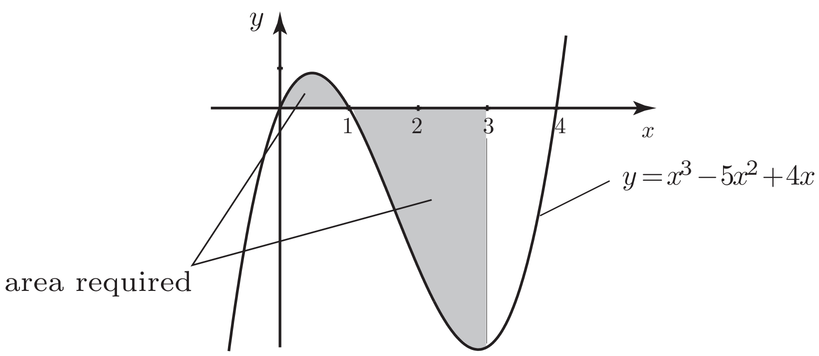


Figure 12

From the graph we see that the required area lies partly above the x-axis (when $0 \leq x \leq 1$) and partly below (when $1 \leq x \leq 3$). So we evaluate the integral in two parts: Firstly:

$$\int_0^1 (x^3 - 5x^2 + 4x)dx = \left[\frac{x^4}{4} - \frac{5x^3}{3} + \frac{4x^2}{2}\right]_0^1 = \left(\frac{1}{4} - \frac{5}{3} + 2\right) - (0) = \frac{7}{12}$$

This is the part of the required area which lies above the x-axis. Secondly:

$$\begin{aligned}\int_1^3 (x^3 - 5x^2 + 4x)dx &= \left[\frac{x^4}{4} - \frac{5x^3}{3} + \frac{4x^2}{2}\right]_1^3 \\ &= \left(\frac{81}{4} - \frac{135}{3} + 18\right) - \left(\frac{1}{4} - \frac{5}{3} + 2\right) = -\frac{22}{3}\end{aligned}$$

This represents the part of the required area which lies below the x-axis. The actual area is $\frac{22}{3}$. Combining the results of the two separate calculations we can find the total area bounded by the curve:

$$\text{area } = \frac{7}{12} + \frac{22}{3} = \frac{95}{12}$$

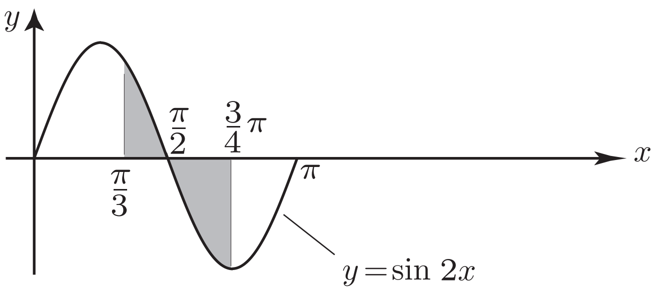
Task

(a) Sketch the graph of $y = \sin 2x$ for $0 \le x \le \pi$.
(b) Find the total area bounded by the curve and the x-axis between $x = \frac{1}{3}\pi$ and $x = \frac{3}{4}\pi$.

(a) Sketch the graph and indicate the required area noting where the graph crosses the x-axis:

Your solution

Answer



(b) Perform the integration in two parts to obtain the required area:

Your solution

Answer

$\displaystyle\int_{\pi/3}^{\pi/2} \sin 2x\,dx = \frac{1}{4}$ and $\displaystyle\int_{\pi/2}^{3\pi/4} \sin 2x dx = -\frac{1}{2}$.

The required area is $\dfrac{1}{4} + \dfrac{1}{2} = \dfrac{3}{4}$.

Exercises

1. Find the total area enclosed between the x-axis and the curve $y = x^3$ between $x = -1$ and $x = 1$.

2. Find the area under $y = \cos 2t$ from $t = 0$ to $t = 0.5$.

3. Find the area enclosed by $y = 4 - x^2$ and the x axis

   (a) from $x = 0$ to $x = 2$, (b) from $x = -2$ to $x = 1$, (c) from $x = 1$ to $x = 3$.

4. Calculate the area enclosed by the curve $y = x^3$ and the line $y = x$.

5. Find the area bounded by $y = \mathrm{e}^x$, the y-axis and the line $x = 2$.

6. Find the area enclosed between $y = x(x-1)(x-2)$ and the x axis.

Answers

1. 0.5 2. 0.4207 3. (a) $\frac{16}{3}$, (b) 9, (c) 4 4. 0.5 5. $\mathrm{e}^2 - 1$ 6. $\frac{1}{2}$

Integration by Parts

13.4

Introduction

Integration by Parts is a technique for integrating products of functions. In this Section you will learn to recognise when it is appropriate to use the technique and have the opportunity to practise using it for finding both definite and indefinite integrals.

Prerequisites

Before starting this Section you should ...

- understand what is meant by definite and indefinite integrals
- be able to use a table of integrals
- be able to differentiate and integrate a range of common functions

Learning Outcomes

On completion you should be able to ...

- decide when it is appropriate to use the method known as integration by parts
- apply the formula for integration by parts to definite and indefinite integrals
- perform integration by parts repeatedly if appropriate

1. Indefinite integration

The technique known as **integration by parts** is used to integrate a product of two functions, such as in these two examples:

(i) $\displaystyle\int \mathrm{e}^{2x}\sin 3x\,dx$ (ii) $\displaystyle\int_0^1 x^3\mathrm{e}^{-2x}\,dx$

Note that in the first example, the integrand is the product of the functions e^{2x} and $\sin 3x$, and in the second example the integrand is the product of the functions x^3 and e^{-2x}. Note also that we can change the order of the terms in the product if we wish and write

(i) $\displaystyle\int (\sin 3x)\,\mathrm{e}^{2x}dx$ (ii) $\displaystyle\int_0^1 \mathrm{e}^{-2x}\,x^3\,dx$

What you must never do is integrate each term in the product separately and then multiply - the integral of a product is not the product of the separate integrals. However, it is often possible to find integrals involving products using the method of integration by parts - you can think of this as a *product rule* for integrals.
The integration by parts formula states:

Key Point 5

Integration by Parts for Indefinite Integrals

For indefinite integrals, given functions $f(x)$ and $g(x)$:

$$\int f\cdot g\,dx = f\cdot\int g\,dx - \int\left(\frac{df}{dx}\cdot\int g dx\right)\,dx$$

Alternatively, given functions u and v:

$$\int u\frac{dv}{dx}dx = u.v - \int v\frac{du}{dx}dx$$

Study the formula carefully and note the following observations. Firstly, to apply the formula we must be able to differentiate the function f to find $\dfrac{df}{dx}$, and we must be able to integrate the function, g. Secondly the formula replaces one integral, the one on the left, with a different integral, that on the far right. The intention is that the latter, whilst it may look more complicated in the formula above, is simpler to evaluate. Consider the following Example:

Example 15

Find the integral of the product of x with $\sin x$; that is, find $\int x \sin x \, dx$.

Solution

Compare the required integral with the formula for integration by parts: we choose

$$f = x \qquad \text{and} \qquad g = \sin x$$

It follows that

$$\frac{df}{dx} = 1 \qquad \text{and} \qquad \int g\,dx = \int \sin x\,dx = -\cos x$$

(When integrating g there is no need to worry about a constant of integration. When you become confident with the method, you may like to think about why this is the case.)

Applying the formula we obtain

$$\begin{aligned} \int x \sin x\,dx &= f \cdot \int g\,dx - \int \left(\frac{df}{dx} \cdot \int g dx \right) dx \\ &= x(-\cos x) - \int 1(-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + c \end{aligned}$$

Task

Find $\displaystyle\int (5x+1)\cos 2x\,dx$.

Let $f = 5x + 1$ and $g = \cos 2x$. Now calculate $\dfrac{df}{dx}$ and $\displaystyle\int g\,dx$:

Your solution

Answer

$\dfrac{df}{dx} = 5$ and $\displaystyle\int \cos 2x\,dx = \frac{1}{2}\sin 2x$.

Substitute these results into the formula for integration by parts and complete the Task:

Your solution

Answer

$(5x+1)(\dfrac{1}{2}\sin 2x) - \displaystyle\int 5(\frac{1}{2}\sin 2x)dx = \frac{1}{2}(5x+1)\sin 2x + \frac{5}{4}\cos 2x + c$

Sometimes it is necessary to apply the formula more than once, as the next Example shows.

Example 16

Find $\displaystyle\int 2x^2\mathrm{e}^{-x}dx$

Solution

We let $f = 2x^2$ and $g = \mathrm{e}^{-x}$. Then $\dfrac{df}{dx} = 4x$ and $\displaystyle\int g dx = -\mathrm{e}^{-x}$

Using the formula for integration by parts we find

$$\int 2x^2\mathrm{e}^{-x}dx = 2x^2(-\mathrm{e}^{-x}) - \int 4x(-\mathrm{e}^{-x})dx = -2x^2\mathrm{e}^{-x} + \int 4x\mathrm{e}^{-x}dx$$

We now need to find $\int 4x\mathrm{e}^{-x}dx$ using integration by parts again. We get

$$\begin{aligned}\int 4xe^{-x}dx &= 4x(-\mathrm{e}^{-x}) - \int 4(-\mathrm{e}^{-x})dx \\ &= -4x\mathrm{e}^{-x} + \int 4\mathrm{e}^{-x}dx = -4x\mathrm{e}^{-x} - 4\mathrm{e}^{-x}\end{aligned}$$

Altogether we have

$$\int 2x^2\mathrm{e}^{-x}dx = -2x^2\mathrm{e}^{-x} - 4x\mathrm{e}^{-x} - 4\mathrm{e}^{-x} + c = -2\mathrm{e}^{-x}(x^2+2x+2) + c$$

Exercises

In some questions below it will be necessary to apply integration by parts more than once.

1. Find (a) $\displaystyle\int x\sin(2x)dx$, (b) $\displaystyle\int t\mathrm{e}^{3t}dt$, (c) $\displaystyle\int x\cos x\,dx$.
2. Find $\displaystyle\int (x+3)\sin x\,dx$.
3. By writing $\ln x$ as $1\times\ln x$ find $\displaystyle\int \ln x\,dx$.
4. Find (a) $\displaystyle\int \tan^{-1}x\,dx$, (b) $\displaystyle\int -7x\cos 3x\,dx$, (c) $\displaystyle\int 5x^2\mathrm{e}^{3x}dx$,
5. Find (a) $\displaystyle\int x\cos kx\,dx$, where k is a constant (b) $\displaystyle\int z^2\cos kz\,dz$, where k is a constant.
6. Find (a) $\displaystyle\int t\mathrm{e}^{-st}dt$ where s is a constant, (b) Find $\displaystyle\int t^2\mathrm{e}^{-st}dt$ where s is a constant.

Answers

1. (a) $\frac{1}{4}\sin 2x - \frac{1}{2}x\cos 2x + c$, (b) $\mathrm{e}^{3t}(\frac{1}{3}t - \frac{1}{9}) + c$, (c) $\cos x + x\sin x + c$

2. $-(x+3)\cos x + \sin x + c$.

3. $x\ln x - x + c$.

4. (a) $x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + c$, (b) $-\frac{7}{9}\cos 3x - \frac{7}{3}x\sin 3x + c$, (c) $\frac{5}{27}\mathrm{e}^{3x}(9x^2 - 6x + 2) + c$,

5. (a) $\dfrac{\cos kx}{k^2} + \dfrac{x\sin kx}{k} + c$, (b) $\dfrac{2z\cos kz}{k^2} + \dfrac{z^2\sin kz}{k} - \dfrac{2\sin kz}{k^3} + c$.

6. (a) $\dfrac{-\mathrm{e}^{-st}(st+1)}{s^2} + c$, (b) $\dfrac{-\mathrm{e}^{-st}(s^2t^2 + 2st + 2)}{s^3} + c$.

2. Definite integration

When dealing with definite integrals the relevant formula is as follows:

Key Point 6

Integration by Parts for Definite Integrals

For definite integrals, given functions $f(x)$ and $g(x)$:

$$\int_a^b f \cdot g\, dx = \left[f \cdot \int g\, dx\right]_a^b - \int_a^b \left(\frac{df}{dx} \cdot \int g dx\right) dx$$

Alternatively, given functions u and v: $\displaystyle\int_a^b u\frac{dv}{dx}dx = \Big[uv\Big]_a^b - \int_a^b v\frac{du}{dx}dx$

Example 17

Find $\displaystyle\int_0^2 x\mathrm{e}^x dx$.

Solution

We let $f = x$ and $g = \mathrm{e}^x$. Then $\dfrac{df}{dx} = 1$ and $\displaystyle\int g\, dx = \mathrm{e}^x$. Using integration by parts we obtain

$$\int_0^2 x\mathrm{e}^x dx = \Big[x\mathrm{e}^x\Big]_0^2 - \int_0^2 1.\mathrm{e}^x dx = 2\mathrm{e}^2 - \Big[\mathrm{e}^x\Big]_0^2 = 2\mathrm{e}^2 - [\mathrm{e}^2 - 1] = \mathrm{e}^2 + 1$$

(or 8.389 to 3 d.p.)

Sometimes it is necessary to apply the formula more than once as the next Example shows.

Example 18

Find the definite integral of $x^2\mathrm{e}^x$ from 0 to 2; that is, find $\displaystyle\int_0^2 x^2\mathrm{e}^x\,dx$.

Solution

We let $f = x^2$ and $g = \mathrm{e}^x$. Then $\dfrac{df}{dx} = 2x$ and $\displaystyle\int g\,dx = \mathrm{e}^x$. Using integration by parts:

$$\int_0^2 x^2\mathrm{e}^x\,dx = \Big[x^2\mathrm{e}^x\Big]_0^2 - \int_0^2 2x\mathrm{e}^x\,dx = 4\mathrm{e}^2 - 2\int_0^2 x\mathrm{e}^x\,dx$$

The remaining integral must be integrated by parts also but we have just done this in the example above. So $\displaystyle\int_0^2 x^2\mathrm{e}^x\,dx = 4\mathrm{e}^2 - 2[\mathrm{e}^2+1] = 2\mathrm{e}^2 - 2 = 12.778$ (3 d.p.)

Task

Find $\displaystyle\int_0^{\pi/4}(4-3x)\sin x\,dx$.

What are your choices for f, g?

Your solution

Answer

Take $f = 4-3x$ and $g = \sin x$.

Now complete the integral:

Your solution

$$\int_0^{\pi/4}(4-3x)\sin x\,dx =$$

Answer

$$\begin{aligned}\int_0^{\pi/4}(4-3x)\sin x\,dx &= \Big[(4-3x)(-\cos x)\Big]_0^{\pi/4} - 3\int_0^{\pi/4}\cos x\,dx\\ &= \Big[(4-3x)(-\cos x)\Big]_0^{\pi/4} - 3\Big[\sin x\Big]_0^{\pi/4}\\ &= 0.716 \text{ to 3 d.p.}\end{aligned}$$

Exercises

1. Evaluate the following: (a) $\displaystyle\int_0^1 x\cos 2x\,dx$, (b) $\displaystyle\int_0^{\pi/2} x\sin 2x\,dx$, (c) $\displaystyle\int_{-1}^1 t\mathrm{e}^{2t}dt$

2. Find $\displaystyle\int_1^2 (x+2)\sin x\,dx$

3. Find $\displaystyle\int_0^1 (x^2-3x+1)\mathrm{e}^x dx$

Answers

1. (a) 0.1006, (b) $\pi/4 = 0.7854$, (c) 1.9488.

2. 3.3533.

3. -0.5634.

Integration by Substitution and Using Partial Fractions

13.5

Introduction

The first technique described here involves making a substitution to simplify an integral. We let a new variable equal a complicated part of the function we are trying to integrate. Choosing the correct substitution often requires experience. This skill develops with practice.

Often the technique of partial fractions can be used to write an algebraic fraction as the sum of simpler fractions. On occasions this means that we can then integrate a complicated algebraic fraction. We shall explore this approach in the second half of the section.

Prerequisites

Before starting this Section you should ...

- be able to find a number of simple definite and indefinite integrals
- be able to use a table of integrals
- be familiar with the technique of expressing an algebraic fraction as the sum of its partial fractions

Learning Outcomes

On completion you should be able to ...

- make simple substitutions in order to find definite and indefinite integrals
- understand the technique used for evaluating integrals of the form $\displaystyle\int \frac{f'(x)}{f(x)}dx$
- use partial fractions to express an algebraic fraction in a simpler form and integrate it

1. Making a substitution

The technique described here involves making a substitution in order to simplify an integral. We let a new variable, u say, equal a more complicated part of the function we are trying to integrate. The choice of which substitution to make often relies upon experience: don't worry if at first you cannot see an appropriate substitution. This skill develops with practice. However, it is not simply a matter of changing the variable - care must be taken with the differential form dx as we shall see. The technique is illustrated in the following Example.

Example 19

Find $\int (3x+5)^6 dx$.

Solution

First look at the function we are trying to integrate: $(3x+5)^6$. It looks quite complicated to integrate. Suppose we introduce a new variable, u, such that $u = 3x+5$. Doing this means that the function we must integrate becomes u^6. Would you not agree that this looks a much simpler function to integrate than $(3x+5)^6$? There is a slight complication however. The new function of u must be integrated with respect to u and not with respect to x. This means that we must take care of the term dx correctly.

Long Method $\quad u = 3x+5 \quad$ so $\quad \dfrac{du}{dx} = 3, \quad$ or $\quad \dfrac{dx}{du} = \dfrac{1}{3}$

Let
$$\begin{aligned} I = \int (3x+5)^6\, dx &= \int u^6\, dx \qquad \text{(substituting for } 3x+5) \\ &= \int u^6 \frac{dx}{du}\, du \quad \text{(to change from } x \text{ to } u) \\ &= \int u^6 \frac{1}{3}.\, du \qquad \text{(substituting for } \frac{dx}{du}) \\ &= \frac{1}{3}\int u^6\, dx = \frac{u^7}{21} + \text{constant} \end{aligned}$$

Short Method $\quad u = 3x+5 \quad$ so $\quad \dfrac{du}{dx} = 3, \quad$ so $\quad dx = \dfrac{1}{3}du$

Let $\quad I = \displaystyle\int (3x+5)^6\, dx = \int u^6\, dx = \int u^6.\frac{1}{3}.\, du = \frac{1}{3}\int u^6\, du = \frac{u^7}{21} + \text{constant}$

To finish off we must rewrite this answer in terms of the original variable x and replace u by $3x+5$:

$$\int (3x+5)^6 dx = \frac{(3x+5)^7}{21} + c$$

In practice the short method is generally used but mathematicians don't like to separate the 'dx' from the 'du' as in the statement '$dx = \frac{1}{3}du$' as it is meaningless mathematically (but it works!). In the future we will use the short method, with apologies to the mathematicians!

Task

By making the substitution $u = \sin x$ find $\int \cos x \sin^2 x \, dx$

You are given the substitution $u = \sin x$. Find $\frac{du}{dx}$:

Your solution

Answer

$\frac{du}{dx} = \cos x$

Now make the substitution, simplify the result, and finally perform the integration:

Your solution

Answer

$\int \cos x \sin^2 x \, dx$ simplifies to $\int u^2 du$. The final answer is $\frac{1}{3}\sin^3 x + c$.

Exercise

Use suitable substitutions to find

(a) $\int (4x+1)^7 dx$ (b) $\int t^2 \sin(t^3+1) dt$ (Hint: you need to simplify $\sin(t^3+1)$)

Answer

(a) $\frac{(4x+1)^8}{32} + c$ (b) $-\frac{\cos(t^3+1)}{3} + c$

2. Substitution and definite integration

If you are dealing with definite integrals (ones with limits of integration) you must be particularly careful when you substitute. Consider the following example.

Example 20

Find the definite integral $\displaystyle\int_2^3 t\sin(t^2)dt$ by making the substitution $u = t^2$.

Solution

Note that if $u = t^2$ then $\dfrac{du}{dt} = 2t$ so that $dt = \dfrac{du}{2t}$. We find

$$\int_{t=2}^{t=3} t\sin(t^2)dt = \int_{t=2}^{t=3} t\sin u\,\frac{du}{2t} = \frac{1}{2}\int_{t=2}^{t=3}\sin u\,du$$

An important point to note is that the limits of integration are limits on the variable t, not u. To emphasise this they have been written explicitly as $t = 2$ and $t = 3$. When we integrate with respect to the variable u, the limits must be written in terms of u. From the substitution $u = t^2$, note that when $t = 2$ then $u = 4$ and when $t = 3$ then $u = 9$ so the integral becomes

$$\frac{1}{2}\int_{u=4}^{u=9}\sin u\,du = \frac{1}{2}\Big[-\cos u\Big]_4^9 = \frac{1}{2}(-\cos 9 + \cos 4) = 0.129 \quad \text{to 3 d.p.}$$

Exercise

Use suitable substitutions to find (a) $\displaystyle\int_1^2 (2x+3)^7 dx$, (b) $\displaystyle\int_0^1 3t^2 e^{t^3} dt$.

Answer

(a) $u = 2x + 3$ is suitable; 3.359×10^5 to 4 sig. figs. (b) 1.718 to 3 d.p.

3. Integrals giving rise to logarithms

Example 21

Find $\displaystyle\int \frac{3x^2+1}{x^3+x+2}dx$

Solution

Let us consider what happens when we make the substitution $z = x^3 + x + 2$. Note that

$$\frac{dz}{dx} = 3x^2 + 1 \qquad \text{so that we can write} \qquad dz = (3x^2+1)dx$$

Then

$$\int \frac{3x^2+1}{x^3+x+2}dx = \int \frac{1}{z}dz = \ln|z| + c = \ln|x^3+x+2|$$

Note that in the last Example, the numerator of the integrand $(3x^2 + 1)$ is the derivative of the denominator $(x^3 + x + 2)$. The result is the logarithm of the denominator. This is a special case of the following rule:

Key Point 7

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Note that it is the modulus of $f(x)$ in the answer.

Task

Write down, purely by inspection, the following integrals:

(a) $\displaystyle\int \frac{1}{x+1}dx$, (b) $\displaystyle\int \frac{2x}{x^2+8}dx$, (c) $\displaystyle\int \frac{1}{x-3}dx$.

Hint: In each case the numerator of the integrand is the derivative of the denominator.

Your solution

(a) (b) (c)

Answer

(a) $\ln|x+1| + c$, (b) $\ln|x^2+8| + c$, (c) $\ln|x-3| + c$

Task

Evaluate the definite integral $\int_2^4 \frac{3t^2+2t}{t^3+t^2+1}dt$.

Your solution

Answer

$$\left[\ln|t^3+t^2+1| \right]_2^4 = \ln 81 - \ln 13 = 1.83$$

Sometimes it is necessary to make slight adjustments to the integrand to obtain a form for which the rule in Key Point 7 is suitable. Consider the next Example.

Example 22

Find the indefinite integral $\int \frac{x^2}{x^3+1}dx$.

Solution

In this Example the derivative of the denominator is $3x^2$ whereas the numerator is just x^2. We adjust the numerator as follows:

$$\int \frac{x^2}{x^3+1}dx = \frac{1}{3}\int \frac{3x^2}{x^3+1}dx \quad \text{and integrate by the rule to get} \quad \tfrac{1}{3}\ln|x^3+1|+c$$

Note that the sort of procedure in the last Example is only possible because we can move **constant factors** through the integral sign. It would be wrong to try to move terms involving the **variable** x in a similar way.

Exercise

Write down the result of finding the following integrals.

(a) $\int \frac{1}{x}dx$, (b) $\int \frac{2t}{t^2+1}dt$, (c) $\int \frac{1}{2x+5}dx$, (d) $\int \frac{2}{3x-2}dx$.

Answer

(a) $\ln|x|+c$, (b) $\ln|t^2+1|+c$, (c) $\frac{1}{2}\ln|2x+5|+c$, (d) $\frac{2}{3}\ln|3x-2|+c$.

4. Integration using partial fractions

Sometimes expressions which at first sight look impossible to integrate using the techniques already met may in fact be integrated by first expressing them as simpler partial fractions, and then using the techniques described earlier in this Section. Consider the following Task.

Task

Express $\dfrac{23-x}{(x-5)(x+4)}$ as the sum of its partial fractions.

Hence find $\displaystyle\int \frac{23-x}{(x-5)(x+4)}dx$

First produce the partial fractions. Write the fraction in the form $\dfrac{A}{x-5}+\dfrac{B}{x+4}$ and find A, B.

Your solution

Answer

$A = 2,\ B = -3$

Now integrate each term separately:

Your solution

$$\int \frac{23-x}{(x-5)(x+4)}dx = \int \frac{A}{x-5}dx + \int \frac{B}{x+4}dx =$$

Answer

$2\ln|x-5| - 3\ln|x+4| + c$

Exercises

By expressing the following in partial fractions, evaluate each integral:

1. $\displaystyle\int \frac{1}{x^3+x}dx$

2. $\displaystyle\int \frac{13x-4}{6x^2-x-2}dx$

3. $\displaystyle\int \frac{1}{(x+1)(x-5)}dx$

4. $\displaystyle\int \frac{2x}{(x-1)^2(x+1)}dx$

Answers

1. $\ln|x| - \dfrac{1}{2}\ln|x^2+1| + c$

2. $\dfrac{3}{2}\ln|2x+1| + \dfrac{2}{3}\ln|3x-2| + c$

3. $\dfrac{1}{6}\ln|x-5| - \dfrac{1}{6}\ln|x+1| + c$

4. $-\dfrac{1}{2}\ln|x+1| + \dfrac{1}{2}\ln|x-1| - \dfrac{1}{x-1} + c$

Integration of Trigonometric Functions

13.6

Introduction

Integrals involving trigonometric functions are commonplace in engineering mathematics. This is especially true when modelling waves and alternating current circuits. When the root-mean-square (rms) value of a waveform, or signal is to be calculated, you will often find this results in an integral of the form

$$\int \sin^2 t \, dt$$

In this Section you will learn how such integrals can be evaluated.

Prerequisites

Before starting this Section you should ...

- be able to find a number of simple definite and indefinite integrals
- be able to use a table of integrals
- be familiar with standard trigonometric identities

Learning Outcomes

On completion you should be able to ...

- use trigonometric identities to write integrands in alternative forms to enable them to be integrated

1. Integration of trigonometric functions

Simple integrals involving trigonometric functions have already been dealt with in Section 13.1. See what you can remember:

Task

Write down the following integrals:

(a) $\int \sin x\, dx$, (b) $\int \cos x\, dx$, (c) $\int \sin 2x\, dx$, (d) $\int \cos 2x\, dx$

Your solution

(a) (b)

(c) (d)

Answer

(a) $-\cos x + c$, (b) $\sin x + c$, (c) $-\dfrac{1}{2}\cos 2x + c$, (d) $\dfrac{1}{2}\sin 2x + c$.

The basic rules from which these results can be derived are summarised here:

Key Point 8

$$\int \sin kx\, dx = -\frac{\cos kx}{k} + c \qquad \int \cos kx\, dx = \frac{\sin kx}{k} + c$$

In engineering applications it is often necessary to integrate functions involving powers of the trigonometric functions such as

$$\int \sin^2 x\, dx \qquad \text{or} \qquad \int \cos^2 \omega t\, dt$$

Note that these integrals cannot be obtained directly from the formulas in Key Point 8 above. However, by making use of trigonometric identities, the integrands can be re-written in an alternative form. It is often not clear which identities are useful and each case needs to be considered individually. Experience and practice are essential. Work through the following Task.

Task

Use the trigonometric identity $\sin^2\theta \equiv \dfrac{1}{2}(1-\cos 2\theta)$ to express the integral $\displaystyle\int \sin^2 x\; dx$ in an alternative form and hence evaluate it.

(a) First use the identity:

Your solution

$$\int \sin^2 x\, dx = \int$$

Answer

The integral can be written $\displaystyle\int \frac{1}{2}(1-\cos 2x)dx$.

Note that the trigonometric identity is used to convert a power of $\sin x$ into a function involving $\cos 2x$ which can be integrated directly using Key Point 8.

(b) Now evaluate the integral:

Your solution

Answer

$\frac{1}{2}\left(x - \frac{1}{2}\sin 2x + c\right) = \frac{1}{2}x - \frac{1}{4}\sin 2x + K$ where $K = c/2$.

Task

Use the trigonometric identity $\sin 2x \equiv 2\sin x\cos x$ to find $\displaystyle\int \sin x \cos x\, dx$

(a) First use the identity:

Your solution

$$\int \sin x \cos x\, dx = \int$$

Answer

The integrand can be written as $\frac{1}{2}\sin 2x$

(b) Now evaluate the integral:

Your solution

Answer

$$\int_0^{2\pi} \sin x\cos x\; dx = \int_0^{2\pi}\frac{1}{2}\sin 2x\, dx = \left[-\frac{1}{4}\cos 2x + c\right]_0^{2\pi} = -\frac{1}{4}\cos 4\pi + \frac{1}{4}\cos 0 = -\frac{1}{4}+\frac{1}{4} = 0$$

This result is one example of what are called **orthogonality relations**.

Engineering Example 3

Magnetic flux

Introduction

The magnitude of the magnetic flux density on the axis of a solenoid, as in Figure 13, can be found by the integral:

$$B = \int_{\beta_1}^{\beta_2} \frac{\mu_0 n I}{2} \sin\beta \; d\beta$$

where μ_0 is the permeability of free space ($\approx 4\pi \times 10^{-7}$ H m^{-1}), n is the number of turns and I is the current.

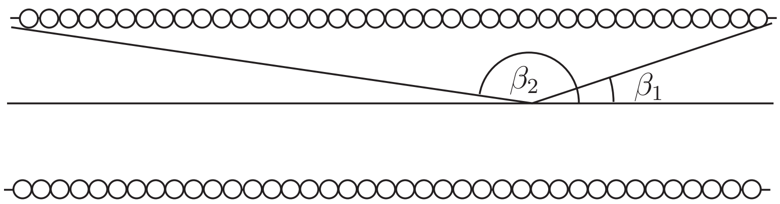


Figure 13: A solenoid and angles defining its extent

Problem in words

Predict the magnetic flux in the middle of a long solenoid.

Mathematical statement of the problem

We assume that the solenoid is so long that $\beta_1 \approx 0$ and $\beta_2 \approx \pi$ so that

$$B = \int_{\beta_1}^{\beta_2} \frac{\mu_0 n I}{2} \sin\beta \; d\beta \approx \int_0^{\pi} \frac{\mu_0 n I}{2} \sin\beta \; d\beta$$

Mathematical analysis

The factor $\dfrac{\mu_0 n I}{2}$ can be taken outside the integral i.e.

$$B = \frac{\mu_0 n I}{2} \int_0^{\pi} \sin\beta \; d\beta = \frac{\mu_0 n I}{2} \Big[-\cos\beta \Big]_0^{\pi} = \frac{\mu_0 n I}{2}(-\cos\pi + \cos 0)$$

$$= \frac{\mu_0 n I}{2}(-(-1) + 1) = \mu_0 n I$$

Interpretation

The magnitude of the magnetic flux density at the midpoint of the axis of a long solenoid is predicted to be approximately $\mu_0 n I$ i.e. proportional to the number of turns and proportional to the current flowing in the solenoid.

2. Orthogonality relations

In general two functions $f(x), g(x)$ are said to be **orthogonal** to each other over an interval $a \leq x \leq b$ if

$$\int_a^b f(x)g(x)\ dx = 0$$

It follows from the previous Task that $\sin x$ and $\cos x$ are orthogonal to each other over the interval $0 \leq x \leq 2\pi$. This is also true over any interval $\alpha \leq x \leq \alpha + 2\pi$ (e.g. $\pi/2 \leq x \leq 5\pi$, or $-\pi \leq x \leq \pi$).

More generally there is a whole set of orthogonality relations involving these trigonometric functions on intervals of length 2π (i.e. over one period of both $\sin x$ and $\cos x$). These relations are useful in connection with a widely used technique in engineering, known as **Fourier analysis** where we represent periodic functions in terms of an infinite series of sines and cosines called a Fourier series. (This subject is covered in HELM 23.)

We shall demonstrate the orthogonality property

$$I_{mn} = \int_0^{2\pi} \sin mx \sin nx\ dx = 0$$

where m and n are integers such that $m \neq n$.

The secret is to use a trigonometric identity to convert the integrand into a form that can be readily integrated.

You may recall the identity

$$\sin A \sin B \equiv \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

It follows, putting $A = mx$ and $B = nx$ that provided $m \neq n$

$$\begin{aligned} I_{mn} &= \frac{1}{2}\int_0^{2\pi} [\cos(m-n)x - \cos(m+n)x]\ dx \\ &= \frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{(m+n)}\right]_0^{2\pi} \\ &= 0 \end{aligned}$$

because $(m-n)$ and $(m+n)$ will be integers and $\sin(\text{integer}\times 2\pi) = 0$. Of course $\sin 0 = 0$.

Why does the case $m = n$ have to be excluded from the analysis? (left to the reader to figure out!)

The corresponding orthogonality relation for cosines

$$J_{mn} = \int_0^{2\pi} \cos mx \cos nx\ dx = 0$$

follows by use of a similar identity to that just used. Here again m and n are integers such that $m \neq n$.

Example 23

Use the identity $\sin A \cos B \equiv \dfrac{1}{2}(\sin(A+B) + \sin(A-B))$ to show that

$$K_{mn} = \int_0^{2\pi} \sin mx \cos nx \; dx = 0 \qquad m \text{ and } n \text{ integers, } m \neq n.$$

Solution

$$\begin{aligned} K_{mn} &= \frac{1}{2}\int_0^{2\pi} [\sin(m+n)x + \sin(m-n)x] \; dx \\ &= \frac{1}{2}\left[-\frac{\cos(m+n)x}{(m+n)} - \frac{\cos(m-n)x}{(m-n)}\right]_0^{2\pi} \\ &= -\frac{1}{2}\left[\frac{\cos(m+n)2\pi - 1}{(m+n)} + \frac{\cos(m-n)2\pi - 1}{(m-n)}\right] = 0 \end{aligned}$$

(recalling that $\cos(\text{integer} \times 2\pi) = 1$)

Task

Derive the orthogonality relation

$$K_{mn} = \int_0^{2\pi} \sin mx \cos nx \; dx = 0 \qquad m \text{ and } n \text{ integers, } m = n$$

Hint: You will need to use a different trigonometric identity to that used in Example 23.

Your solution

Answer

$K_{mn} = \displaystyle\int_0^{2\pi} \sin mx \cos mx \; dx$

Putting $m = n \neq 0$, and then using the identity $\sin 2A \equiv 2 \sin A \cos A$ we get

$$\begin{aligned} K_{mm} &= \int_0^{2\pi} \sin mx \cos mx \; dx \\ &= \frac{1}{2}\int_0^{2\pi} \sin 2mx \; dx \\ &= \frac{1}{2}\left[-\frac{\cos 2mx}{2m}\right]_0^{2\pi} = -\frac{1}{4m}(\cos 4m\pi - \cos 0) = -\frac{1}{4m}(1-1) = 0 \end{aligned}$$

Putting $m = n = 0$ gives $K_{00} = \dfrac{1}{2}\displaystyle\int_0^{2\pi} \sin 0 \cos 0 \; dx = 0.$

Note that the particular case $m = n = 1$ was considered earlier in this Section.

3. Reduction formulae

You have seen earlier in this Workbook how to integrate $\sin x$ and $\sin^2 x$ (which is $(\sin x)^2$). Applications sometimes arise which involve integrating higher powers of $\sin x$ or $\cos x$. It is possible, as we now show, to obtain a **reduction formula** to aid in this Task.

Task

Given $I_n = \displaystyle\int \sin^n(x) \; dx$ write down the integrals represented by I_2, I_3, I_{10}

Your solution

$I_2 =$ $\qquad$ $I_3 =$ $\qquad$ $I_{10} =$

Answer

$$I_2 = \int \sin^2 x \; dx \qquad I_3 = \int \sin^3 x \; dx \qquad I_{10} = \int \sin^{10} x \; dx$$

To obtain a reduction formula for I_n we write

$$\sin^n x = \sin^{n-1}(x) \sin x$$

and use integration by parts.

Task

In the notation used earlier in this Workbook for integration by parts (Key Point 5, page 31) put $f = \sin^{n-1} x$ and $g = \sin x$ and evaluate $\dfrac{df}{dx}$ and $\displaystyle\int g\ dx$.

Your solution

Answer

$\dfrac{df}{dx} = (n-1)\sin^{n-2}x\ \cos x$ (using the chain rule of differentiation),

$\displaystyle\int g\ dx = \int \sin x\ dx = -\cos x$

Now use the integration by parts formula on $\displaystyle\int \sin^{n-1} x \sin x\ dx$. [Do not attempt to evaluate the second integral that you obtain.]

Your solution

Answer

$$\begin{aligned}\int \sin^{n-1} x \sin x\ dx &= \sin^{n-1}(x)\int g\ dx - \int \frac{df}{dx}\int g\ dx \\ &= \sin^{n-1}(x)(-\cos x) + (n-1)\int \sin^{n-2} x \cos^2 x\ dx\end{aligned}$$

We now need to evaluate $\displaystyle\int \sin^{n-2} x \cos^2 x dx$. Putting $\cos^2 x = 1 - \sin^2 x$ this integral becomes:

$$\int \sin^{n-2}(x)\ dx - \int \sin^n(x)\ dx$$

But this is expressible as $I_{n-2} - I_n$ so finally, using this and the result from the last Task we have

$I_n = \displaystyle\int \sin^{n-1}(x)\sin x\ dx = \sin^{n-1}(x)(-\cos x) + (n-1)(I_{n-2} - I_n)$

from which we get Key Point 9:

Key Point 9

Reduction Formula

Given $I_n = \int \sin^n x dx$

$$I_n = -\frac{1}{n}\sin^{n-1}(x)\cos x + \frac{n-1}{n}I_{n-2}$$

This is our **reduction formula** for I_n. It enables us, for example, to evaluate I_6 in terms of I_4, then I_4 in terms of I_2 and I_2 in terms of I_0 where

$$I_0 = \int \sin^0 x\ dx = \int 1\ dx = x.$$

Task

Use the reduction formula in Key Point 9 with $n = 2$ to find I_2.

Your solution

Answer

$$\begin{aligned} I_2 &= -\frac{1}{2}[\sin x \cos x] + \frac{1}{2}I_0 \\ &= -\frac{1}{2}[\frac{1}{2}\sin 2x] + \frac{x}{2} + c \\ \text{i.e.} \qquad \int \sin^2 x\ dx &= -\frac{1}{4}\sin 2x + \frac{x}{2} + c \end{aligned}$$

as obtained earlier by a different technique.

Task

Use the reduction formula in Key Point 9 to obtain $I_6 = \int \sin^6 x\, dx$.

Firstly obtain I_6 in terms of I_4, then I_4 in terms of I_2:

Your solution

Answer

Using Key Point 9 with $n = 6$ gives $I_6 = -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}I_4$.

Then, using Key Point 9 again with $n = 4$, gives $I_4 = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}I_2$

Now substitute for I_2 from the previous Task to obtain I_4 and hence I_6.

Your solution

Answer

$$I_4 = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{16}\sin 2x + \frac{3}{8}x + \text{ constant}$$

$$\therefore \qquad I_6 = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{32}\sin 2x + \frac{5}{16}x + \text{ constant}$$

Definite integrals can also be readily evaluated using the reduction formula in Key Point 9. For example,

$$I_n = \int_0^{\pi/2} \sin^n x\, dx \quad \text{so} \quad I_{n-2} = \int_0^{\pi/2} \sin^{n-2} x\, dx$$

We obtain, immediately

$$I_n = \frac{1}{n}\Big[-\sin^{n-1}(x)\cos x\Big]_0^{\pi/2} + \frac{n-1}{n}I_{n-2}$$

or, since $\cos\frac{\pi}{2} = \sin 0 = 0$, $\qquad I_n = \frac{(n-1)}{n}I_{n-2}$

This simple easy-to-use formula is well known and is called **Wallis' formula**.

Key Point 10

Reduction Formula - Wallis' Formula

Given $I_n = \int_0^{\pi/2} \sin^n x\, dx$ or $I_n = \int_0^{\pi/2} \cos^n x\, dx$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

Task

If $I_n = \int_0^{\pi/2} \sin^n x\ dx$ calculate I_1 and then use Wallis' formula, without further integration, to obtain I_3 and I_5.

Your solution

Answer

$I_1 = \int_0^{\pi/2} \sin x\ dx = \Big[-\cos x \Big]_0^{\pi/2} = 1$

Then using Wallis' formula with $n = 3$ and $n = 5$ respectively

$$I_3 = \int_0^{\pi/2} \sin^3 x\ dx = \frac{2}{3} I_1 = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$I_5 = \int_0^{\pi/2} \sin^5 x\ dx = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$$

Task

The total power P of an antenna is given by

$$P = \int_0^{\pi} \frac{\eta L^2 I^2 \pi}{4\lambda^2} \sin^3\theta \; d\theta$$

where η, λ, I are constants as is the length L of antenna. Using the reduction formula for $\int \sin^n x \; dx$ in Key Point 9, obtain P.

Your solution

Answer

Ignoring the constants for the moment, consider

$$I_3 = \int_0^{\pi} \sin^3\theta \; d\theta \text{ which we will reduce to } I_1 \text{ and evaluate.}$$

$$I_1 = \int_0^{\pi} \sin\theta \; d\theta = \Big[-\cos\theta \Big]_0^{\pi} = 2$$

so by the reduction formula with $n = 3$

$$I_3 = \frac{1}{3}\Big[-\sin^2 x \cos x \Big]_0^{\pi} + \frac{2}{3} I_1 = 0 + \frac{2}{3} \times 2 = \frac{4}{3}$$

We now consider the actual integral with all the constants.

Hence $P = \dfrac{\eta L^2 I^2 \pi}{4\lambda^2} \displaystyle\int_0^{\pi} \sin^3\theta \; d\theta = \dfrac{\eta L^2 I^2 \pi}{4\lambda^2} \times \dfrac{4}{3}$, so $P = \eta \dfrac{L^2 I^2 \pi}{3\lambda^2}$.

A similar reduction formula to that in Key Point 9 can be obtained for $\int \cos^n x \; dx$ (see Exercise 5 at the end of this Workbook). In particular if

$$J_n = \int_0^{\pi/2} \cos^n x \; dx \quad \text{then} \quad J_n = \frac{(n-1)}{n} J_{n-2}$$

i.e. Wallis' formula is the same for $\cos^n x$ as for $\sin^n x$.

4. Harder trigonometric integrals

The following seemingly innocent integrals are examples, important in engineering, of trigonometric integrals that **cannot** be evaluated as **indefinite** integrals:

(a) $\displaystyle\int \sin(x^2)\,dx$ and $\displaystyle\int \cos(x^2)\,dx$ These are called Fresnel integrals.

(b) $\displaystyle\int \frac{\sin x}{x}\,dx$ This is called the Sine integral.

Definite integrals of this type, which are what normally arise in applications, have to be evaluated by **approximate numerical methods**.

Fresnel integrals with limits arise in wave and antenna theory and the Sine integral with limits in filter theory.

It is useful sometimes to be able to visualize the definite integral. For example consider

$$F(t) = \int_0^t \frac{\sin x}{x}\,dx \qquad t > 0$$

Clearly, $F(0) = \displaystyle\int_0^0 \frac{\sin x}{x}\,dx = 0$. Recall the graph of $\dfrac{\sin x}{x}$ against x, $x > 0$:

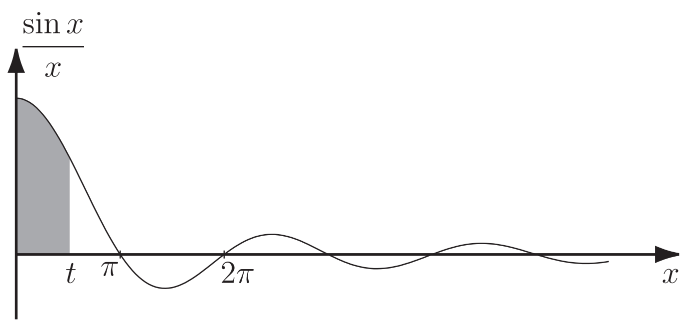


Figure 14

For any positive value of t, $F(t)$ is the shaded area shown (the area interpretation of a definite integral was covered earlier in this Workbook). As t increases from 0 to π, it follows that $F(t)$ increases from 0 to a maximum value

$$F(\pi) = \int_0^\pi \frac{\sin x}{x}\,dx$$

whose value could be determined numerically (it is actually about 1.85). As t further increases from π to 2π the value of $F(t)$ will decrease to a local minimum at 2π because the $\dfrac{\sin x}{x}$ curve is below the x-axis between π and 2π. Note that the area below the curve **is** considered to be negative in this application.

Continuing to argue in this way we can obtain the shape of the $F(t)$ graph in Figure 15: (can you see why the oscillations decrease in amplitude?)

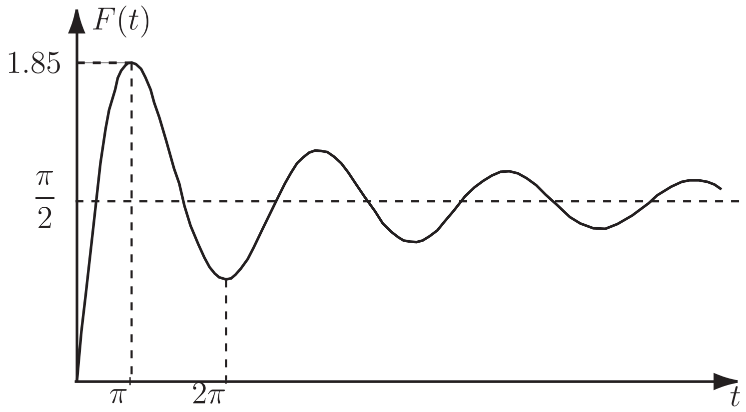


Figure 15

The result $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ is clearly illustrated in the graph (you are not expected to know how this result is obtained). Methods for solving such problems are dealt with in HELM 31.

Exercises

You will need to refer to a Table of Trigonometric Identities to answer these questions.

1. Find (a) $\displaystyle\int \cos^2 x dx$ (b) $\displaystyle\int_0^{\pi/2} \cos^2 t dt$ (c) $\displaystyle\int (\cos^2\theta + \sin^2\theta) d\theta$

2. Use the identity $\sin(A+B) + \sin(A-B) \equiv 2\sin A\cos B$ to find $\displaystyle\int \sin 3x \cos 2x dx$

3. Find $\displaystyle\int (1 + \tan^2 x) dx$.

4. The mean square value of a function $f(t)$ over the interval $t = a$ to $t = b$ is defined to be

$$\frac{1}{b-a}\int_a^b (f(t))^2 dt$$

Find the mean square value of $f(t) = \sin t$ over the interval $t = 0$ to $t = 2\pi$.

5. (a) Show that the reduction formula for $J_n = \displaystyle\int \cos^n x\ dx$ is

$$J_n = \frac{1}{n}\cos^{n-1}(x)\sin x + \frac{(n-1)}{n}J_{n-2}$$

(b) Using the reduction formula in (a) show that

$$\int \cos^5 x\ dx = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x$$

(c) Show that if $J_n = \displaystyle\int_0^{\pi/2} \cos^n x\ dx$, then $J_n = \left(\dfrac{n-1}{n}\right) J_{n-2}$ (Wallis' formula).

(d) Using Wallis' formula show that $\displaystyle\int_0^{\pi/2} \cos^6 x\ dx = \frac{5}{32}\pi$.

Answers

1. (a) $\frac{1}{2}x + \frac{1}{4}\sin 2x + c$ (b) $\pi/4$ (c) $\theta + c$.
2. $-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$.
3. $\tan x + c$.
4. $\frac{1}{2}$.